

# ON THE JORDAN STRUCTURE OF HOLOMORPHIC MATRICES

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**ABSTRACT.** Let  $X$  be an open subset of  $\mathbb{C}^N$ , and let  $A$  be an  $n \times n$  matrix of holomorphic functions on  $X$ . We call a point  $\xi \in X$  **Jordan stable** for  $A$  if  $\xi$  is not a splitting point of the eigenvalues of  $A$  and, moreover, there is a neighborhood  $U$  of  $\xi$  such that, for each  $1 \leq k \leq n$ , the number of Jordan blocks of size  $k$  in the Jordan normal forms of  $A(\zeta)$  is the same for all  $\zeta \in U$ . H. Baumgärtel [B4, S 3.4] proved that there is a nowhere dense closed analytic subset of  $X$ , which contains all points of  $X$  which are not Jordan stable for  $A$ . We give a new proof of this result. This proof has the advantage that the result can be obtained in a more precise form, and with some estimates. Also, this proof applies to arbitrary, possibly non-smooth, complex spaces  $X$ .

## 1. INTRODUCTION

Let  $X$  be an open subset of  $\mathbb{C}^N$ , and let  $A$  be an  $n \times n$  matrix of holomorphic functions on  $X$ .

A point  $\xi \in X$  is called a **splitting point of the eigenvalues** of  $A$  if, for each neighborhood  $U \subseteq X$  of  $\xi$ , there is a point  $\zeta \in U$  such that  $A(\zeta)$  has more eigenvalues than  $A(\xi)$  (see Lemma 4.2 for an equivalent definition). It is well-known (cp. Remark 3.4) that the set of all splitting points of the eigenvalues of  $A$  is a nowhere dense closed analytic subset of  $X$ .<sup>1</sup>

We call a point  $\xi \in X$  **Jordan stable** for  $A$  if  $\xi$  is not a splitting point of the eigenvalues of  $A$  and, moreover, there is a neighborhood  $U$  of  $\xi$  such that, for each  $1 \leq k \leq n$ , the number of Jordan blocks of size  $k$  in the Jordan normal forms of  $A(\zeta)$  is the same for all  $\zeta \in U$  (see Definition 5.4 for equivalent conditions). Denote by  $\text{Jst } A$  the set of all Jordan stable points of  $A$ .

H. Baumgärtel proved that  $X \setminus \text{Jst } A$  is contained in some nowhere dense closed analytic subset of  $X$ , see [B1], [B2, Kap. V, §7], [B4, 5.7] for  $N = 1$ , and [B3], [B4, S 3.4] for arbitrary  $N$ .

In the present paper, we give a new proof for Baumgärtel's theorem, and also for the analyticity of the set of splitting points of the eigenvalues. These proofs have the advantage that the results can be obtained in a more precise form, and with some estimates. For example (Theorem 5.5):

The set  $X \setminus \text{Jst } A$  is not only *contained* in a nowhere dense closed analytic subset of  $X$ , but it is itself such a set. Moreover, there exist finitely many holomorphic functions  $f_1, \dots, f_n : X \rightarrow \mathbb{C}$  such that

$$X \setminus \text{Jst } A = \{f_1 = \dots = f_\ell = 0\}$$

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<sup>1</sup> $Y \subseteq X$  is called a closed analytic subset of  $X$  if, for each point  $\xi \in X$ , there exist a neighborhood  $U \subseteq X$  of  $\xi$  and holomorphic functions  $f_1, \dots, f_\ell$  on  $U$  such that  $Y \cap U = \{f_1 = \dots = f_\ell = 0\}$ . For  $N = 1$  this means that  $Y$  is a closed discrete subset of  $X$ .

and, for some constants  $K, k \in \mathbb{N}^*$  depending only on  $n$  (and not on  $X$  and  $A$ ),

$$|f(\zeta)| \leq K(1 + \|A(\zeta)\|)^k \quad \text{for all } \zeta \in X.$$

This implies:

- If  $X$  is the open unit disk in  $\mathbb{C}$  and  $A$  is bounded, then  $X \setminus \text{Jst } A$  satisfies the Blaschke condition.
- If  $X = \mathbb{C}^N$  and the elements of  $A$  are holomorphic polynomials, then  $X \setminus \text{Jst } A$  is affine algebraic. For  $N = 1$  this means that  $X \setminus \text{Jst } A$  is finite.

If  $X$  has a  $\mathcal{C}^0$  boundary, corresponding results are obtained for functions which admit a continuous extension to the boundary of  $X$  (Section 6).

Also, our proof applies to arbitrary, possibly non-smooth, complex spaces  $X$ .

## 2. NOTATION

$\mathbb{N}$  denotes the set of natural numbers including 0.  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ .

If  $n, m \in \mathbb{N}^*$ , then  $\text{Mat}(n \times m, \mathbb{C})$  denotes the space of complex  $n \times m$  matrices ( $n$  rows and  $m$  columns), and  $\text{GL}(n, \mathbb{C})$  denotes the group of invertible elements of  $\text{Mat}(n \times n, \mathbb{C})$ .

For  $\Phi \in \text{Mat}(n \times m, \mathbb{C})$ , we denote by  $\text{Ker } \Phi$ ,  $\text{Im } \Phi$ ,  $\text{rank } \Phi$  and  $\|\Phi\|$  the kernel, the image, the rank and the operator norm (as a linear map between the Euclidean spaces  $\mathbb{C}^m$  and  $\mathbb{C}^n$ ) of  $\Phi$ , respectively.

The unit matrix in  $\text{Mat}(n \times n, \mathbb{C})$  will be denoted by  $I_n$  or simply by  $I$ . If  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$  and  $\lambda \in \mathbb{C}$ , then, instead of  $\lambda I_n - \Phi$  we write also  $\lambda - \Phi$ .

By a complex space we always mean a *reduced* complex space in the sense of, e.g., [GR], which is the same as an *analytic* space in the sense of, e.g., [L]. For example, each complex manifold and each analytic subset of a complex manifold is a complex space.

By an *irreducible* complex space we mean a *globally* irreducible complex space, i.e., a complex space, for which the manifold of smooth points is connected, see, e.g., [L, Ch. V.4.5] or [GR, Ch. 9, §1]. For example, each connected complex manifold is an irreducible complex space.

If we say “ $\lambda_1, \dots, \lambda_m$  are the eigenvalues of a matrix” (or the zeros of a polynomial), then we mean this always not counting multiplicities (hence, then  $\lambda_i \neq \lambda_j$  if  $i \neq j$ ).

## 3. SPLITTING POINTS OF THE ZEROS OF MONIC POLYNOMIALS

**3.1. Definition.** By a complex polynomial we mean a function  $p : \mathbb{C} \rightarrow \mathbb{C}$  of the form  $p(\lambda) = p_0 + p_1\lambda + \dots + p_n\lambda^n$ , where  $n \in \mathbb{N}$  and  $p_0, \dots, p_n \in \mathbb{C}$ . If  $p_n = 1$ ,  $p$  is called **monic**. The map from  $\mathbb{C}$  to  $\mathbb{C}$  which is identically zero will be called the **zero polynomial**.

If  $n \in \mathbb{N}$ , then we denote by  $\mathcal{P}_n$  the complex vector space, which consists of all complex polynomials of degree  $\leq n$  and the zero polynomial.

Now let  $X$  be a topological space,  $n \in \mathbb{N}^*$ , and  $P : X \rightarrow \mathcal{P}_n$  a continuous map, all values of which are monic and of degree  $n$ . Then  $\xi \in X$  is called a **splitting point of the zeros of  $P$**  if, for each neighborhood  $U$  of  $\xi$ , there exists  $\zeta \in U$  such that  $P(\zeta)$  has more zeros than  $P(\xi)$  (not counting multiplicities).

Equivalently, one can define the *non-splitting* points, using the following well-known lemma. For completeness, we supply a proof.

**3.2. Lemma.** *Let  $X$  be a topological space,  $n \in \mathbb{N}^*$ , and  $P : X \rightarrow \mathcal{P}_n$  a continuous map, all values of which are monic and of degree  $n$ . Let  $\xi \in X$ , let  $w_1, \dots, w_m$  be the zeros of  $P(\xi)$ , and let  $n_j$  be the order of  $w_j$  as a zero of  $P(\xi)$ . Then  $\xi$  is not a splitting point of the zeros of  $P$  if and only if the following condition is satisfied:*

*If  $U$  is a sufficiently small connected open neighborhood of  $\xi$ , then there are uniquely determined continuous functions  $\lambda_1, \dots, \lambda_m : U \rightarrow \mathbb{C}$ , which are holomorphic if  $X$  is a complex space and  $P$  is holomorphic, such that*

- $\lambda_j(\xi) = w_j$  for  $1 \leq j \leq m$ ,
- for each  $\zeta \in U$ ,  $\lambda_1(\zeta), \dots, \lambda_m(\zeta)$  are the zeros of  $P(\zeta)$ <sup>2</sup>, and the orders of these zeros are  $n_1, \dots, n_m$ , respectively.

*Proof.* It is clear that the condition is sufficient.

Assume that  $\xi$  is not a splitting point of the zeros of  $P$ .

Then, by definition, there is a neighborhood  $U$  of  $\xi$  such that

$$(3.1) \quad m \geq \text{the numbers of zeros of } P(\zeta), \quad \text{for all } \zeta \in U.$$

Choose  $\varepsilon > 0$  such that the disks

$$(3.2) \quad \mathbb{D}_j := \{z \in \mathbb{C} \mid |z - w_j| < \varepsilon\}, \quad 1 \leq j \leq m,$$

are pairwise disjoint. Since  $P$  is continuous and  $P(\xi)(z) \neq 0$  for  $z \in (\partial\mathbb{D}_1 \cup \dots \cup \partial\mathbb{D}_m)$ , shrinking  $U$ , we can achieve that  $|P(\zeta)(z) - P(\xi)(z)| < |P(\xi)(z)|$ , for all  $\zeta \in U$  and  $z \in (\partial\mathbb{D}_1 \cup \dots \cup \partial\mathbb{D}_m)$ . Then it follows from Rouché's theorem that, for each  $\zeta \in U$  and each  $1 \leq j \leq m$ , counting multiplicities,  $P(\zeta)$  has exactly  $n_j$  zeros in  $\mathbb{D}_j$ . Since the disks  $\mathbb{D}_1, \dots, \mathbb{D}_m$  are pairwise disjoint and by (3.1), this implies:

- for all  $\zeta \in U$  and  $1 \leq j \leq m$ ,  $P(\zeta)$  has exactly one zero in  $\mathbb{D}_j$ ,  $\lambda_j(\zeta)$ , where  $n_j$  is the multiplicity of this zero,
- for all  $\zeta \in U$ ,  $\lambda_k(\zeta) \neq \lambda_j(\zeta)$  if  $1 \leq k, j \leq m$  with  $k \neq j$ ,
- for all  $\zeta \in U$ ,  $\lambda_1(\zeta), \dots, \lambda_m(\zeta)$  are the zeros of  $P(\zeta)$ .

It remains to prove that the so defined functions  $\lambda_1, \dots, \lambda_m : U \rightarrow \mathbb{C}$  are continuous (resp. holomorphic) in  $U$ .

Let  $\zeta \in U$ . Since  $\lambda_j(\zeta)$  is the only zero of  $P(\zeta)$  in  $\mathbb{D}_j \cup \partial\mathbb{D}$  and the order of this zero is  $n_j$ , the function

$$z \mapsto z \frac{P(\zeta)'(z)}{P(\zeta)(z)},$$

where  $P(\zeta)'$  is the complex derivative of  $P(\zeta)$ , has exactly one singularity in  $\mathbb{D}_j \cup \partial\mathbb{D}_j$ , namely  $\lambda_j(\zeta)$ , and the residuum of this singularity is  $n_j \lambda_j(\zeta)$ . Hence

$$(3.3) \quad \lambda_j(\zeta) = \frac{1}{n_j 2\pi i} \int_{\partial\mathbb{D}_j} z \frac{P(\zeta)'(z)}{P(\zeta)(z)} dz \quad \text{for } 1 \leq j \leq m.$$

This formula shows that  $\lambda_1, \dots, \lambda_m$  are continuous, for  $P$  is continuous, and, moreover, holomorphic if  $X$  is a complex space and  $P$  is holomorphic.  $\square$

The following theorem is contained, e.g., in the lemma at the beginning of Chapter V, §7.1 of [L], applied to the projection

$$\{(\zeta, \lambda) \in X \times \mathbb{C} \mid P(\zeta)(\lambda) = 0\} \longrightarrow X.$$

<sup>2</sup>In particular,  $\lambda_i(\zeta) \neq \lambda_j(\zeta)$  if  $i \neq j$ , according to our convention at the end of Section 2.

**3.3. Theorem.** *Let  $X$  be a complex space and let  $P : X \rightarrow \mathcal{P}_n$  be a holomorphic map, all values of which are of degree  $n$  and monic. Then the splitting points of the zeros of  $P$  form a nowhere dense closed analytic subset of  $X$ .*

**3.4. Remark.** If  $X$  is smooth, there are many sources for this in the literature, see, e.g., [GF, Ch. III, Satz 6.5 and Satz 6.12], [FG, Ch. III, Theorems 4.3 and 4.6], [B3], [B4, S3.1]. There, the fact is used that  $P$  can be written as a finite product

$$(3.4) \quad P = \omega_1^{r_1} \cdot \dots \cdot \omega_\ell^{r_\ell},$$

where  $r_i \in \mathbb{N}^*$ , each  $\omega_i$  is a monic polynomial with coefficients from  $\mathcal{O}(X)$  of positive degree<sup>3</sup>, each  $\omega_i$  is prime as an element of the monoid of all monic polynomials with coefficients from  $\mathcal{O}(X)$ , and  $\omega_i \neq \omega_j$  if  $i \neq j$ . Then it is proved that the discriminant of the polynomial  $\omega_1(\zeta) \cdot \dots \cdot \omega_\ell(\zeta)$ ,  $\Delta$ , does not identically vanish, and  $\{\Delta = 0\}$  is the set of splitting points of the zeros of  $P$ .

Note that this proof also shows that the set of splitting points of the zeros of  $P$ , at each point of this set, is of codimension 1 in  $X$ .

In this section we give a new proof of Theorem 3.3, which results in a more precise result with estimates. In this proof we do not use the factorization (3.4) (also not for the smooth part of  $X$ ). The main tool of our proof is the following lemma, which is known (see, e.g., [KN, §2, 1, VII] or [GH, Theorem 0.1]). For completeness, we give a proof.

**3.5. Lemma.** <sup>4</sup> *Let  $p$  be a monic complex polynomial of degree  $n$ ,  $n \in \mathbb{N}^*$ . Denote by  $\mathcal{P}_{-1}$  the space which consists only of the zero polynomial. Let*

$$\Phi : \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1} \rightarrow \mathcal{P}_{2n-2}$$

*be the linear map defined by*

$$\Phi(s, q) = ps - p'q \quad \text{for } (s, q) \in \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1},$$

*where  $p'$  denotes the complex derivative of  $p$ . Further, let  $m$  be the number of zeros of  $p$  (not counting multiplicities). Then*

$$(3.5) \quad \text{rank } \Phi = n + m - 1.$$

*Proof.* Let  $\lambda_1, \dots, \lambda_m$  be the zeros of  $p$ , and  $k_j$  the order of  $\lambda_j$  as a zero of  $p$ . Since  $p$  is of degree  $n$  and monic, then  $k_1 + \dots + k_m = n$  and

$$p(\lambda) = (\lambda - \lambda_1)^{k_1} \dots (\lambda - \lambda_m)^{k_m}, \quad \lambda \in \mathbb{C}.$$

Set  $q_0(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_m)$  and  $s_0(\lambda) = \sum_{j=1}^m k_j (\lambda - \lambda_1) \dots \widehat{\lambda_j} \dots (\lambda - \lambda_m)$ . Then

$$(3.6) \quad ps_0 = p'q_0.$$

Next we prove that

$$(3.7) \quad \text{Ker } \Phi = \left\{ (s_0 a, q_0 a) \mid a \in \mathcal{P}_{n-1-m} \right\}.$$

<sup>3</sup>By that we mean that, for some  $k_i \in \mathbb{N}^*$ ,  $\omega_i$  is a holomorphic map from  $X$  to  $\mathcal{P}_{k_i}$  all values of which are of degree  $k_i$  and monic.

<sup>4</sup>Formulas (3.14) and (3.15) below show that  $\pm \det \Phi$  is the discriminant of  $p$  (see, e.g., [vdW, §35]). Therefore, this lemma in particular contains the well-known fact that  $p$  has no multiple zeros if and only if its discriminant is different from zero.

*Proof of “ $\supseteq$ ”:* For  $m = n$  this is trivial. Let  $1 \leq m \leq n - 1$  and  $a \in \mathcal{P}_{n-1-m}$ . Since  $s_0$  is of degree  $m - 1$  and  $q_0$  of degree  $m$ , then  $(s_0a, q_0a) \in \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}$ , and by (3.6),  $\Phi_p(s_0a, q_0a) = (ps_0 - p'q_0)a = 0$ .

*Proof of “ $\subseteq$ ”:* Let  $(s, q) \in \text{Ker } \Phi$ , i.e.,  $s \in \mathcal{P}_{n-2}$ ,  $q \in \mathcal{P}_{n-1}$  and

$$(3.8) \quad ps = p'q.$$

Then each  $\lambda_j$  is a zero of order  $\geq k_j$  of  $p'q$ . Since the order of  $\lambda_j$  as a zero of  $p'$  is  $< k_j$  (for  $k_j = 1$ , by this we mean that  $p'(\lambda_j) \neq 0$ ), it follows that each  $\lambda_j$  is a zero of  $q$ . Hence,  $q$  is of the form

$$(3.9) \quad q = q_0a,$$

where  $a$  is some complex polynomial (possibly,  $a \equiv 0$ ). If  $a \neq 0$ , from (3.9) it follows that  $\deg a = \deg q - \deg q_0$ . Since  $\deg q_0 = m$  and  $\deg q \leq n - 1$ , this implies that  $\deg a \leq n - 1 - m$ , i.e.,

$$(3.10) \quad a \in \mathcal{P}_{n-1-m}.$$

If  $a \equiv 0$ , (3.10) holds trivially. Moreover, by (3.6), (3.9) and (3.8),

$$ps_0a = p'q_0a = p'q = ps.$$

As  $p \neq 0$ , this implies that  $s = s_0a$ . Together with (3.9) and (3.10) this proves that  $(s, q)$  belongs to the right hand side of (3.7).

So (3.7) is proved. Now we consider the linear map

$$\begin{aligned} \Psi : \mathcal{P}_{n-1-m} &\longrightarrow \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1} \\ a &\longmapsto (s_0a, q_0a). \end{aligned}$$

Since  $s_0 \neq 0$  and  $q_0 \neq 0$ , this map is injective. Hence

$$\dim \text{Im } \Psi = \dim \mathcal{P}_{n-1-m} = n - m.$$

As, by (3.7),  $\text{Im } \Psi = \text{Ker } \Phi$ , it follows that

$$\dim \text{Ker } \Phi = \dim \text{Im } \Psi = n - m.$$

As  $\text{rank } \Phi = 2n - 1 - \dim \text{Ker } \Phi$ , this proves (3.5).  $\square$

Also we use a simple fact on the jump behavior of the rank of a continuous matrix.

**3.6. Definition.** Let  $X$  be a topological space, and  $M : X \rightarrow \text{Mat}(n \times m, \mathbb{C})$  a continuous map. A point  $\xi \in X$  will be called a **jump point** of rank  $M$  if, for each neighborhood  $U$  of  $\xi$ , there is a point  $\zeta \in U$  such that  $\text{rank } M(\zeta) > \text{rank } M(\xi)$ .

Since the function  $X \ni \zeta \mapsto \text{rank } M(\zeta)$  is lower semicontinuous, then  $\xi \in X$  is not a jump point of rank  $M$  if and only if there is a neighborhood  $U$  of  $\xi$  such that the map

$$U \ni \zeta \longmapsto \text{rank } M(\zeta)$$

is constant.

**3.7. Lemma.** Let  $X$  be an irreducible complex space,  $M : X \rightarrow \text{Mat}(n \times m, \mathbb{C})$  holomorphic, and

$$r_{\max} := \max_{\zeta \in X} \text{rank } M(\zeta).$$

Let  $f_1, \dots, f_\ell$  be the minors of order  $r_{\max}$  of  $M$ . Then  $\{f_1 = \dots = f_\ell = 0\}$  is the set of jump points of rank  $M$ .

*Proof.* Let  $\xi \in X$  be given. First assume that  $\xi$  is a jump point of rank  $M$ . Then, in particular,  $\text{rank } M(\xi) < r_{\max}$ . Hence  $f_1(\xi) = \dots = f_\ell(\xi) = 0$ .

Now we assume that  $\xi$  is not a jump point of rank  $M$ . We have to prove that then  $\text{rank } M(\xi) = r_{\max}$ . Assume to the contrary that  $\text{rank } M(\xi) < r_{\max}$ . Since  $\xi$  is not a jump point of rank  $M$ , then there is a neighborhood  $U$  of  $\xi$  such that  $\text{rank } M(\zeta) \leq \text{rank } M(\xi)$  for all  $\zeta \in U$ . On the other hand, as  $M$  is continuous, there is a neighborhood  $V$  of  $\xi$  such that  $\text{rank } M(\zeta) \geq \text{rank } M(\xi)$  for all  $\zeta \in V$ . Hence,  $\text{rank } M(\zeta) = \text{rank } M(\xi) < r_{\max}$  for all  $\zeta \in U \cap V$ , which means that  $f_1 = \dots = f_\ell = 0$  on  $U \cap V$ . Since  $X$  is irreducible and, hence, the manifold of smooth points of  $X$  is connected and everywhere dense in  $X$ , and since the functions  $f_j$  are holomorphic, it follows that  $f_1 = \dots = f_\ell = 0$  on all of  $X$ , which is impossible by definition of  $r_{\max}$ .  $\square$

Now we are ready to prove Theorem 3.3. Actually, we prove the more precise

**3.8. Theorem.** *Let  $X$  be a complex space,  $n \in \mathbb{N}^*$ , and let  $P : X \rightarrow \mathcal{P}_n$  be a holomorphic map, all values of which are monic and of degree  $n$ . Denote by  $\text{split } P$  the set of splitting points of the zeros of  $P$ .*

*Then  $\text{split } P$  is a nowhere dense closed analytic subset of  $X$ .*

*Moreover, if  $X$  is irreducible and  $\text{split } P \neq \emptyset$ , and if  $P_0(\zeta), \dots, P_n(\zeta)$  are the coefficients of  $P(\zeta)$ , then there exist finitely many holomorphic functions  $h_1, \dots, h_\ell : X \rightarrow \mathbb{C}$ , each of which is a finite sum of finite products of some of the coefficients  $P_0, \dots, P_{n-1}$ , such that*

$$(3.11) \quad \text{split } P = \{h_1 = \dots = h_\ell = 0\},$$

*and*

$$(3.12) \quad |h_j(\zeta)| \leq (2n)^{4n} \max_{0 \leq \mu \leq n-1} |P_\mu(\zeta)|^{2n} \quad \text{for all } \zeta \in X \text{ and } 1 \leq j \leq \ell.$$

*Proof.* If  $\text{split } P = \emptyset$ , the claim of the theorem is trivial. Therefore, we may assume that  $\text{split } P \neq \emptyset$ .

First, moreover assume that  $X$  is irreducible.

Let  $L(\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}, \mathcal{P}_{2n-2})$  be the space a linear maps from  $\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}$  to  $\mathcal{P}_{2n-2}$ , and let

$$\Phi : X \rightarrow L(\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}, \mathcal{P}_{2n-2})$$

be the holomorphic map defined by

$$(3.13) \quad \Phi(\zeta)(s, q) = P(\zeta)s - P(\zeta)'q, \quad \zeta \in X, \quad (s, q) \in \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1},$$

where  $P(\zeta)'$  is the complex derivative of the polynomial  $P(\zeta)$ .

For  $\lambda \in \mathbb{C}$ , we define

$$\begin{aligned} u_j(\lambda) &= \begin{cases} (\lambda^j, 0) & \text{for } j = 0, \dots, n-1, \\ (0, \lambda^{j-n}) & \text{for } j = n, \dots, 2n-2, \end{cases} \\ v_j(\lambda) &= \lambda^j \quad \text{for } j = 0, \dots, 2n-2. \end{aligned}$$

Then  $u_0, \dots, u_{2n-2}$  is a basis of  $\mathcal{P}_{n-1} \oplus \mathcal{P}_{n-2}$  and  $v_0, \dots, v_{2n-2}$  is a basis of  $\mathcal{P}_{2n-2}$ . Let  $M = (M_{ij})_{i,j=0}^{2n-2}$  be the corresponding representation matrix of  $\Phi$ , i.e.,

$$\Phi(\zeta)u_j = \sum_{i=0}^{2n-2} M_{ij}(\zeta)v_i \quad \text{for } \zeta \in X \text{ and } 0 \leq j \leq 2n-2.$$

Then, by (3.13), for  $0 \leq j \leq n-1$ , we have

$$(\Phi(\zeta)u_j)(\lambda) = \sum_{i=0}^n P_i(\zeta)\lambda^{i+j} = \sum_{i=j}^{n+j} P_{i-j}(\zeta)v_i(\lambda),$$

and, for  $n \leq j \leq 2n-2$ , we have

$$(\Phi(\zeta)u_j)(\lambda) = -\sum_{i=1}^n iP_i(\zeta)\lambda^{i-1+j-n} = \sum_{i=j-n}^{j-1} (j-i-n-1)P_{i-j+n+1}(\zeta)v_i(\lambda).$$

Hence, for  $0 \leq j \leq n-1$ , we have

$$(3.14) \quad M_{ij}(\zeta) = \begin{cases} P_{i-j}(\zeta) & \text{if } j \leq i \leq n+j, \\ 0 & \text{otherwise,} \end{cases}$$

and, for  $n \leq j \leq 2n-2$ , we have

$$(3.15) \quad M_{ij}(\zeta) = \begin{cases} (j-i-n-1)P_{i-j+n+1}(\zeta) & \text{if } j-n \leq i \leq j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$r_{\max} := \max_{\zeta \in X} \text{rank } M(\zeta),$$

and let  $h_1, \dots, h_\ell$  be the minors of order  $r_{\max}$  of  $M$  which do not vanish identically on  $X$  (by definition of  $r_{\max}$ , there are such minors). Then, by Lemma 3.7,  $\{h_1 = \dots = h_\ell = 0\}$  is the set of jump points of  $\text{rank } M$ . Since, by Lemma 3.5,

$$\text{rank } M(\zeta) = \text{the number of zeros of } P(\zeta) + n - 1, \quad \text{for all } \zeta \in X,$$

this proves (3.11).

As the manifold of smooth points of  $X$  is a connected ( $X$  is irreducible) and dense subset of  $X$  and the functions  $h_j$  do not identically vanish on  $X$ , (3.11) in particular shows that  $\text{split } P$  is a nowhere dense closed analytic subset of  $X$ .

Since  $\text{split } P \neq \emptyset$ , from (3.11) we moreover see that none of the functions  $h_j$  is zero free. Therefore, it follows (3.14) and (3.15) that each  $h_j$  is a finite sum of products of some of the coefficients  $P_1, \dots, P_{n-1}$  (recall that  $P_n \equiv 1$ ) and that

$$|h_j(\zeta)| \leq r_{\max}! \left( n \max_{0 \leq \mu \leq n-1} |P_\mu(\zeta)| \right)^{r_{\max}} \quad \text{for } 1 \leq j \leq \ell,$$

which implies (3.12).

Now we consider the general case. By the global decomposition theorem for complex spaces (see, e.g., [L, V.4.6] or [GR, Ch. 9, §2.2]), there is a locally finite covering  $\{X_i\}_{i \in I}$  of  $X$  such that each  $X_i$  is an irreducible closed analytic subset of  $X$ . Then, clearly,

$$\text{split } P = \bigcup_{i \in I} \left( X_i \cap \text{split } (P|_{X_i}) \right),$$

and, as already proved, each  $X_i \cap \text{split } (P|_{X_i})$  is a nowhere dense analytic subset of  $X_i$ . Since the covering  $\{X_i\}_{i \in I}$  is locally finite, this proves that  $\text{split } P$  is a nowhere dense analytic subset of  $X$ .  $\square$

**3.9. Remark.** Our proof of Theorem 3.8 does not show that, at each point of  $\text{split } P$  which is a smooth point of  $X$ ,  $\text{split } P$  is of codimension 1 in  $X$  (in distinction to the well-known proof outlined in Remark 3.4). An advantage of this proof is that it shows that, in the irreducible case,  $\text{split } P$  can be defined by finite sums of finite

products of the coefficients of  $P$ , which satisfy estimate (3.12). This implies, for example:

- If  $X = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$  and the coefficients of  $P$  are bounded, then split  $P$  satisfies the Blaschke condition.
- If  $X = \mathbb{C}^N$ , and the coefficients of  $P$  are holomorphic polynomials, then split  $P$  is defined by finitely many holomorphic polynomials. For  $N = 1$  this means that split  $P$  is finite (which is well-known from the theory of algebraic functions).

#### 4. SPLITTING POINTS OF THE EIGENVALUES OF A MATRIX FUNCTION

**4.1. Definition.** Let  $X$  be a topological space, and  $A : X \rightarrow \text{Mat}(n \times n; \mathbb{C})$  continuous. A point  $\xi \in X$  is called a **splitting point of the eigenvalues of  $A$**  if, for each neighborhood  $U$  of  $\xi$ , there exists  $\zeta \in U$  such that  $A(\zeta)$  has more eigenvalues than  $A(\xi)$  (not counting multiplicities).

Since, for each  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$ , the eigenvalues of  $\Phi$  are the zeros of the characteristic polynomial  $\det(\lambda - \Phi)$ ,  $\lambda \in \mathbb{C}$ , which is of degree  $n$  and monic, from Lemma 3.2 we immediately obtain the following characterization of the *non-splitting* points of the eigenvalues of a matrix.

**4.2. Lemma.** Let  $X$  be a topological space,  $n \in \mathbb{N}^*$ , and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  a continuous map. Let  $\xi \in X$ , let  $w_1, \dots, w_m$  be the eigenvalues of  $A(\xi)$ , and let  $n_j$  be the algebraic multiplicity of  $w_j$ .<sup>5</sup> Then  $\xi$  is not a splitting point of the eigenvalues of  $A$  if and only if the following condition is satisfied:

If  $U$  is a sufficiently small connected open neighborhood of  $\xi$ , then there are uniquely determined continuous functions  $\lambda_1, \dots, \lambda_m : U \rightarrow \mathbb{C}$ , which are holomorphic if  $X$  is a complex space and  $A$  is holomorphic, such that

- $\lambda_j(\xi) = w_j$  for  $1 \leq j \leq m$ ,
- for each  $\zeta \in U$ ,  $\lambda_1(\zeta), \dots, \lambda_m(\zeta)$  are the eigenvalues of  $A(\zeta)$ , where  $n_j$  is the algebraic multiplicity  $\lambda_j(\zeta)$ .

**4.3. Theorem.** Let  $X$  be a complex space and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  holomorphic. Denote by  $\text{split } A$  the set of splitting points of the eigenvalues of  $A$ .

Then  $\text{split } A$  is a nowhere dense closed analytic subset of  $X$ .

Moreover, if  $X$  is irreducible and  $\text{split } A \neq \emptyset$ , then there exist finitely many holomorphic functions  $h_1, \dots, h_\ell : X \rightarrow \mathbb{C}$ , each of which is a finite sum of finite products of elements of  $A$ , such that

$$(4.1) \quad \text{split } A = \{h_1 = \dots = h_\ell = 0\},$$

and

$$(4.2) \quad |h_j(\zeta)| \leq (2n)^{6n^2} \|A(\zeta)\|^{2n^2} \quad \text{for all } \zeta \in X \text{ and } 1 \leq j \leq \ell.$$

*Proof.* Let  $P(\zeta)(\lambda) := \det(\lambda - A(\zeta))$ , for  $\zeta \in X$  and  $\lambda \in \mathbb{C}$ , and let  $\text{split } P$  be the set of splitting points of the zeros of  $P$ . Since the eigenvalues of  $A$  are the zeros of  $P$ , then

$$\text{split } A = \text{split } P.$$

Therefore, by Theorem 3.8,  $\text{split } A$  is a nowhere dense analytic subset of  $X$ .

Now we assume that  $X$  is irreducible and  $\text{split } A \neq \emptyset$ . Let  $P_1(\zeta), \dots, P_n(\zeta)$  be the coefficients of  $P(\zeta)$ . Then, again by Theorem 3.8, there exist finitely many holomorphic functions  $h_1, \dots, h_\ell : X \rightarrow \mathbb{C}$ , each of which is a finite sum of finite

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<sup>5</sup>i.e., the order as a zero of the characteristic polynomial of  $A(\xi)$ .



products of some of the coefficients  $P_0, \dots, P_{n-1}$  and, hence, a finite sum of finite products of elements of  $A$ , such that

$$(4.3) \quad \text{split } P = \{h_1 = \dots = h_\ell = 0\}$$

and

$$(4.4) \quad |h_j(\zeta)| \leq (2n)^{4n} \max_{0 \leq \mu \leq n-1} |P_\mu(\zeta)|^{2n} \quad \text{for all } \zeta \in X \text{ and } 1 \leq j \leq \ell.$$

Since  $\text{split } A = \text{split } P$ , then (4.1) follows from (4.3).

If  $0 \leq \mu \leq n-1$ , then  $|P_\mu(\zeta)| \leq n! \|A(\zeta)\|^n$  for all  $\zeta \in X$ . Therefore it follows from (4.4) that

$$|h_j(\zeta)| \leq (2n)^{4n} \left( n! \|A(\zeta)\|^n \right)^{2n} \quad \text{for all } \zeta \in X \text{ and } 1 \leq j \leq \ell,$$

which implies (4.2).  $\square$

**4.4. Remark.** According to the end of Remark 3.4, the claim of Theroem 4.3 can be completed by the statement that, at each point of  $\text{split } A$  which is a smooth point of  $X$ ,  $\text{split } A$  is of codimension 1 in  $X$ .

## 5. JORDAN STABLE POINTS

**5.1. Definition.** As usual, by a **Jordan block** we mean a matrix of the form  $\lambda I_\ell + (\delta_{i,j-1})_{i,j=1}^\ell$ , where  $\delta_{ij}$  is the Kronecker symbol,  $\lambda \in \mathbb{C}$  (the eigenvalue of the Jordan block) and  $\ell \in \mathbb{N}^*$  (the size of the Jordan block).

If  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$  and  $\lambda_1, \dots, \lambda_m$  are the eigenvalues of  $\Phi$ , then, for  $\ell \in \mathbb{N}^*$ , we denote by  $\vartheta_\ell(\Phi, \lambda_j)$  the number of Jordan blocks of size  $\ell$  of the eigenvalue  $\lambda_j$  in the Jordan normal forms of  $\Phi$ , and set

$$\vartheta_\ell(\Phi, \bullet) = \sum_{j=1}^m \vartheta_\ell(\Phi, \lambda_j).$$

Further, then we define

$$\Theta_\Phi = (\lambda_1 - \Phi) \cdot \dots \cdot (\lambda_m - \Phi),$$

which is correct, for the matrices  $\lambda_1 - \Phi, \dots, \lambda_m - \Phi$  pairwise commute.

**5.2. Lemma.** Let  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$ ,  $\lambda_1, \dots, \lambda_m$  the eigenvalues of  $\Phi$ , and  $n_j$  the algebraic multiplicity of  $\lambda_j$  (i.e., its order as a zero of the characteristic polynomial of  $\Phi$ ). Then

$$(5.1) \quad \text{rank}(\lambda_j - \Phi)^k = n - n_j \quad \text{for } k \geq n_j \quad \text{and } 1 \leq j \leq m$$

$$(5.2) \quad \Theta_\Phi^k = 0 \quad \text{for } k \geq n,$$

$$(5.3) \quad \text{rank } \Theta_\Phi^k = n - nm + \text{rank}(\lambda_1 - \Phi)^k + \dots + \text{rank}(\lambda_m - \Phi)^k \quad \text{if } 1 \leq k \leq n-1,$$

$$(5.4) \quad \text{rank } \Theta_\Phi^k = n - \sum_{\ell=1}^k \ell \vartheta_\ell(\Phi, \bullet) - k \sum_{\ell=k+1}^n \vartheta_\ell(\Phi, \bullet) \quad \text{if } 1 \leq k \leq n-1,$$

$$(5.5) \quad \vartheta_k(\Phi, \lambda_j) = \text{rank}(\lambda_j - \Phi)^{k-1} + \text{rank}(\lambda_j - \Phi)^{k+1} - 2 \text{rank}(\lambda_j - \Phi)^k \quad \text{if } 1 \leq k \leq n \text{ and } 1 \leq j \leq m,$$

where  $(\lambda_j - \Phi)^0 := I_n$ .

For completeness, we give a proof of this lemma, although the relations collected there (and in its proof) are well-known, possibly, in somewhat different formulations, see, e.g., [B2, Kap. II, §8.4] or [B4, 2.9.4].

*Proof.* First note that, if, for some  $1 \leq j \leq m$ ,  $J$  is a Jordan block of size  $\ell$  and with eigenvalue  $\lambda_j$ , then

$$(5.6) \quad \begin{aligned} \text{rank}(\lambda_j - J)^k &= \ell - k \quad \text{for } 0 \leq k \leq \ell - 1, \\ (\lambda_j - J)^\ell &= 0, \\ \lambda_i - J &\in \text{GL}(\ell, \mathbb{C}) \quad \text{for all } 1 \leq i \leq m \text{ with } i \neq j. \end{aligned}$$

Denote by  $E_j$  the algebraic eigenspace of  $\lambda_j$ , i.e.,  $E_j := \text{Ker}(\lambda_j - \Phi)^{n_j}$ . Then each  $E_j$  is an invariant subspace of each  $\lambda_i - \Phi$ , and, since  $\Phi$  is similar to a matrix in Jordan normal form, it follows from (5.6) that

$$(5.7) \quad \mathbb{C}^n = E_1 \oplus \dots \oplus E_m, \quad \text{and} \quad n_j = \dim E_j \quad \text{for } 1 \leq j \leq m,$$

$\lambda_i - \Phi$  maps  $E_j$  isomorphically onto itself if  $i \neq j$ ,

$$(5.8) \quad \text{Ker}(\lambda_j - \Phi)^k = E_j \quad \text{for } k \geq n_j \quad \text{and} \quad 1 \leq j \leq m,$$

$$(5.9) \quad \dim \text{Ker}(\lambda_j - \Phi)^k = \sum_{\ell=1}^k \ell \vartheta_\ell(\Phi, \lambda_j) + k \sum_{\ell=k+1}^{n_j} \vartheta_\ell(\Phi, \lambda_j) \quad \text{for } 1 \leq j \leq m \text{ and } k \in \mathbb{N}^*,$$

and (taking into account that the matrices  $\lambda_j - \Phi$  pairwise commute), for all  $k \in \mathbb{N}^*$ ,

$$(5.10) \quad \text{Ker} \Theta_\Phi^k = \text{Ker}(\lambda_1 - \Phi)^k \oplus \dots \oplus \text{Ker}(\lambda_m - \Phi)^k,$$

$$(5.11) \quad \dim \text{Ker} \Theta_\Phi^k = \dim \text{Ker}(\lambda_1 - \Phi)^k + \dots + \dim \text{Ker}(\lambda_m - \Phi)^k.$$

From (5.9) and (5.11) together, we obtain

$$(5.12) \quad \dim \text{Ker} \Theta_\Phi^k = \sum_{\ell=1}^k \ell \vartheta_\ell(\Phi, \bullet) + k \sum_{\ell=k+1}^n \vartheta_\ell(\Phi, \bullet), \quad k \in \mathbb{N}^*.$$

Now: (5.1) follows from (5.7) and (5.8); (5.2) follows from (5.7), (5.8) and (5.10); (5.3) follows from (5.11); (5.4) follows from (5.12).

To prove (5.5), we first note that (5.9) holds also for  $k = 0$  – then both sides are zero. Hence, for  $k \in \mathbb{N}^*$  and  $1 \leq j \leq m$ ,

$$\begin{aligned} & \dim \text{Ker}(\lambda_j - \Phi)^k - \dim \text{Ker}(\lambda_j - \Phi)^{k-1} \\ &= \left( \sum_{\ell=1}^k \ell \vartheta_\ell(\Phi, \lambda_j) - \sum_{\ell=1}^{k-1} \ell \vartheta_\ell(\Phi, \lambda_j) \right) + \left( k \sum_{\ell=k+1}^{n_j} \vartheta_\ell(\Phi, \lambda_j) - (k-1) \sum_{\ell=k}^{n_j} \vartheta_\ell(\Phi, \lambda_j) \right) \\ &= k \vartheta_k(\Phi, \lambda_j) + \sum_{\ell=k+1}^{n_j} \vartheta_\ell(\Phi, \lambda_j) - (k-1) \vartheta_k(\Phi, \lambda_j) = \sum_{\ell=k}^{n_j} \vartheta_\ell(\Phi, \lambda_j) \end{aligned}$$

and, therefore,

$$\vartheta_k(\Phi, \lambda_j) = 2 \dim \text{Ker}(\lambda_j - \Phi)^k - \dim \text{Ker}(\lambda_j - \Phi)^{k-1} - \dim \text{Ker}(\lambda_j - \Phi)^{k+1},$$

which implies (5.5).  $\square$

**5.3. Lemma.** *Let  $X$  be a topological space,  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  a continuous map, and  $\xi$  a point in  $X$  which is not a splitting point of the eigenvalues of  $A$ , and let  $w_1, \dots, w_m$  be the eigenvalues of  $A(\xi)$ , with the algebraic multiplicities  $n_1, \dots, n_m$ , respectively. Take a sufficiently small connected open neighborhood  $U$  of  $\xi$  such that (by Lemma 4.2) there are uniquely determined continuous functions  $\lambda_1, \dots, \lambda_m : U \rightarrow \mathbb{C}$ , which are holomorphic if  $X$  is a complex space and  $A$  is holomorphic, such that*

- $\lambda_j(\xi) = w_j$  for  $1 \leq j \leq m$ , and
- for each  $\zeta \in U$ ,  $\lambda_1(\zeta), \dots, \lambda_m(\zeta)$  are the eigenvalues of  $A(\zeta)$ , where  $n_j$  is the algebraic multiplicity of  $\lambda_j(\zeta)$ .

*Then the following conditions are equivalent.*

- (i) *There exists a neighborhood  $V \subseteq U$  of  $\xi$  such that the map*

$$(5.13) \quad V \ni \zeta \mapsto \vartheta_\ell(A(\zeta), \lambda_j(\zeta))$$

*is constant if  $1 \leq j \leq m$  and  $1 \leq \ell \leq n$ .*

- (ii) *There exists a neighborhood  $V \subseteq U$  of  $\xi$  such that the map*

$$(5.14) \quad V \ni \zeta \mapsto \vartheta_\ell(A(\zeta), \bullet)$$

*is constant if  $1 \leq \ell \leq n$ .*

- (iii) *There exists a neighborhood  $V \subseteq U$  of  $\xi$  such that the map*

$$(5.15) \quad V \ni \zeta \mapsto \text{rank}(\Theta_{A(\zeta)})^k$$

*is constant if  $1 \leq k \leq n - 1$ .*

- (iv) *There exists a neighborhood  $V \subseteq U$  of  $\xi$  such that the map*

$$(5.16) \quad V \ni \zeta \mapsto \text{rank}(\lambda_j(\zeta) - A(\zeta))^k$$

*is constant if  $1 \leq k \leq n - 1$ .*

- (v) *There exists a neighborhood  $V \subseteq U$  of  $\xi$  and a continuous map  $T : V \rightarrow \text{GL}(n, \mathbb{C})$ , which is holomorphic if  $X$  is a complex space and  $A$  is holomorphic, such that  $T(\zeta)^{-1}A(\zeta)T(\zeta)$  is in Jordan normal form for all  $\zeta \in V$ .*

If  $X$  is a domain in  $\mathbb{C}$  and  $A$  is holomorphic, the equivalence of conditions (i), (ii) and (v) is due to G. P. A. Thiesse [T].

*Proof.* The equivalence of (i) - (iv) follows from Lemma 5.2. Indeed:

(i)  $\Rightarrow$  (ii) is trivial.

(ii)  $\Rightarrow$  (iii) follows from (5.4).

To prove that (iii)  $\Rightarrow$  (iv), we note that, by (5.3),

$$\text{rank}(\Theta_{A(\zeta)})^k = \text{const} + \text{rank}(\lambda_1(\zeta) - A(\zeta))^k + \dots + \text{rank}(\lambda_m(\zeta) - A(\zeta))^k$$

for all  $\zeta \in U$  and  $1 \leq k \leq n - 1$ , and observe that the functions on the right hand side of this relation are lower semicontinuous in  $\zeta$  (since the rank of a continuous matrix function is always lower semicontinuous). If the left hand side is constant for  $\zeta$  in some neighborhood  $V$  of  $\xi$ , this is possible only if also the functions on the right hand side are constant for  $\zeta \in V$ .

(iv)  $\Rightarrow$  (i) follows from (5.5).

Moreover, it is clear that (v)  $\Rightarrow$  (i). To complete the proof of the lemma, therefore is sufficient to prove that (i)  $\Rightarrow$  (v).

Assume (i) is satisfied.

For each  $1 \leq j \leq m$ , choose a matrix  $N_j \in \text{Mat}(n_j \times n_j, \mathbb{C})$ , which is a block diagonal matrix with Jordan blocks on the diagonal, each of which has the eigenvalue 0, and such that, for each  $\ell \in \mathbb{N}^*$ , exactly  $\vartheta_\ell(A(\xi), \lambda_j(\xi))$  of them are of size  $\ell$ . Since the algebraic multiplicity of  $\lambda_j(\xi)$  is  $n_j$ , then  $N_j$  is an  $n_j \times n_j$  matrix. Hence, for each  $1 \leq j \leq m$  and each  $\zeta \in V$ ,  $\lambda_j(\zeta)I_{n_j} + N_j$  is a Jordan block with eigenvalue  $\lambda_j(\zeta)$  and of size  $n_j$ . Let  $J : U \rightarrow \text{Mat}(n \times n, \mathbb{C})$  (note that  $n_1 + \dots + n_m = n$ ) be the map such that  $J(\zeta)$ ,  $\zeta \in U$ , is the block diagonal matrix with the diagonal

$$\lambda_1(\zeta)I_{n_1} + N_1, \dots, \lambda_m(\zeta)I_{n_m} + N_m.$$

Since the functions  $\lambda_j$  are continuous, and holomorphic if  $A$  is holomorphic, then  $J$  is continuous, and holomorphic if  $A$  is holomorphic. Moreover, since condition (i) is satisfied, there is a neighborhood  $V \subseteq U$  of  $\xi$  such that,

$$\vartheta_\ell(A(\zeta), \lambda_j(\zeta)) = \vartheta_\ell(A(\xi), \lambda_j(\xi)) \quad \text{for all } \zeta \in V,$$

which means that, for each  $\zeta \in V$ ,  $J(\zeta)$  is a Jordan normal form of  $A(\zeta)$ , i.e., there exists a matrix  $\Theta_\zeta \in \text{GL}(n, \mathbb{C})$  with

$$(5.17) \quad \Theta_\zeta J(\zeta) \Theta_\zeta^{-1} = A(\zeta).$$

Now let  $\text{End}(\text{Mat}(n \times n, \mathbb{C}))$  be the space of linear endomorphisms of the complex vector space  $\text{Mat}(n \times n, \mathbb{C})$ . Following an idea of W. Wasow [W], we consider the continuous (and holomorphic if  $A$  is holomorphic) map  $\varphi : V \rightarrow \text{End}(\text{Mat}(n \times n, \mathbb{C}))$  defined by

$$\varphi(\zeta)\Phi = \Phi A(\zeta) - J(\zeta)\Phi, \quad \zeta \in V, \quad \Phi \in \text{Mat}(n \times n, \mathbb{C}).$$

We claim that the map

$$(5.18) \quad V \ni \zeta \longmapsto \dim \text{Ker } \varphi(\zeta)$$

is constant. Indeed, by definition of  $\varphi$  and by (5.17), for all  $\zeta \in V$ ,

$$\begin{aligned} \text{Ker } \varphi(\zeta) &= \left\{ \Phi \in \text{Mat}(n \times n, \mathbb{C}) \mid \Phi A(\zeta) = J(\zeta)\Phi \right\} \\ &= \left\{ \Phi \in \text{Mat}(n \times n, \mathbb{C}) \mid \Phi \Theta_\zeta J(\zeta) = J(\zeta)\Phi \Theta_\zeta \right\} \\ &= \left\{ \Phi \in \text{Mat}(n \times n, \mathbb{C}) \mid \Phi J(\zeta) = J(\zeta)\Phi \right\} \Theta_\zeta^{-1}. \end{aligned}$$

In particular, for all  $\zeta \in V$ ,

$$(5.19) \quad \dim \text{Ker } \varphi(\zeta) = \dim \left\{ \Phi \in \text{Mat}(n \times n, \mathbb{C}) \mid \Phi J(\zeta) = J(\zeta)\Phi \right\}.$$

Since  $\lambda_i(\zeta) \neq \lambda_j(\zeta)$  if  $i \neq j$ , it follows from [Ga, Ch. VIII, §1] that, for all  $\zeta \in W$ , a matrix belongs to the space on the right hand side of (5.19) if and only if it is a block diagonal matrix with a diagonal of the form  $\Lambda_1, \dots, \Lambda_m$ , where  $\Lambda_j$  belongs to the space

$$\begin{aligned} &\left\{ \Phi \in \text{Mat}(n_j \times n_j, \mathbb{C}) \mid \Phi(\lambda_j(\zeta)I_{n_j} + N_j) = (\lambda_j(\zeta)I_{n_j} + N_j)\Phi \right\} \\ &= \left\{ \Phi \in \text{Mat}(n_j \times n_j, \mathbb{C}) \mid \Phi N_j = N_j \Phi \right\}. \end{aligned}$$

Since the latter space is independent of  $\zeta$ , this means that (5.18) is constant.

Since  $\varphi$  is continuous, and holomorphic if  $A$  is holomorphic, the constancy of (5.18) means that the family  $\{\text{Ker } \varphi(\zeta)\}_{\zeta \in V}$  is a sub-vector bundle of the product

bundle  $W \times \text{Mat}(n \times n, \mathbb{C})$ , which is holomorphic if  $A$  is holomorphic (see, e.g., [W, Lemma 1] or [Sh, Corollary 2]).

Therefore, through each point in this sub-vector bundle goes a local continuous (resp. holomorphic) section. Since, by (5.17),  $(\xi, \Theta_\xi^{-1})$  is such a point, it follows that there is a neighborhood  $V$  of  $\xi$  and a continuous (resp. holomorphic) map  $S : V \rightarrow \text{Mat}(n \times n, \mathbb{C})$  with  $S(\xi) = \Theta_\xi^{-1}$  and  $S(\zeta)A(\zeta) = J(\zeta)S(\zeta)$  for all  $\zeta \in V$ . Since  $\Theta_\xi^{-1}$  is invertible, shrinking  $V$ , we may achieve that moreover  $S(\zeta) \in \text{GL}(n, \mathbb{C})$  for all  $\zeta \in V$ . It remains to set  $T(\zeta) = S(\zeta)^{-1}$  for  $\zeta \in V$ .  $\square$

**5.4. Definition.** Let  $X$  be a topological space, and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  a continuous map. A point  $\xi \in X$  is called **Jordan stable** for  $A$  if  $\xi$  is not a splitting point of the eigenvalues of  $A$  and the equivalent conditions (i) - (v) in Lemma 5.3 are satisfied.

If  $G$  is a domain in some  $\mathbb{C}^N$  and  $A : G \rightarrow \text{Mat}(n \times n, \mathbb{C})$  is holomorphic, H. Baumgärtel proved that there exists a nowhere dense analytic subset  $B$  of  $G$ , which contains the splitting points of  $A$ , such that all points of  $G \setminus B$  are Jordan stable for  $A$  (he proved that condition (v) in Lemma 5.3 is satisfied), see [B1], [B2, Kap. V, §7] and [B4, 5.7] if  $N = 1$ , and [B3] and [B4, S 3.4] for arbitrary  $N$ .

In the present section we give a new proof of Baumgärtel's theorem, which gives the following more precise and more general

**5.5. Theorem.** *Let  $X$  be a complex space, and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  holomorphic. Denote by  $\text{Jst } A$  the set of Jordan stable points of  $A$ .*

*Then  $X \setminus \text{Jst } A$  is a nowhere dense closed analytic subset of  $X$ .*

*Moreover, if  $X$  is irreducible and normal<sup>6</sup>, and if  $\text{Jst } A \neq X$ , then there exist finitely many holomorphic functions  $h_1, \dots, h_\ell : X \rightarrow \mathbb{C}$ , each of which is a finite sum of finite products of elements of  $A$ , such that*

$$(5.20) \quad X \setminus \text{Jst } A = \{h_1 = \dots = h_\ell = 0\}$$

and

$$(5.21) \quad |h_j(\zeta)| \leq (2n)^{2n^4} \|A(\zeta)\|^{2n^4} \quad \text{for all } \zeta \in X \text{ and } 1 \leq j \leq \ell.$$

*Proof.* For  $\text{Jst } A = \emptyset$ , the claim of the theorem is trivial. Therefore we may assume that  $\text{Jst } A \neq \emptyset$ .

We first consider the case when  $X$  is normal and irreducible.

Let  $\text{split } A$  be the set of splitting points of the eigenvalues of  $A$ , and let  $X^0$  be the manifold of smooth points of  $X$ . Since  $X^0$  is connected ( $X$  is irreducible) and dense in  $X$ , and  $\text{split } A$  is a nowhere dense analytic subset of  $X$  (Theorem 4.3),  $X \setminus \text{split } A$  is connected.

Consider the map

$$(5.22) \quad X \setminus \text{split } A \ni \zeta \longmapsto \Theta_{A(\zeta)}.$$

By Lemma 4.2, for each  $\xi \in X \setminus \text{split } A$ , we have an open neighborhood  $U_\xi \subseteq X \setminus \text{split } A$  of  $\xi$  and holomorphic functions  $\lambda_1^{(\xi)}, \dots, \lambda_m^{(\xi)} : U_\xi \rightarrow \mathbb{C}$  such that, for all  $\zeta \in U_\xi$ ,  $\lambda_1^{(\xi)}(\zeta), \dots, \lambda_m^{(\xi)}(\zeta)$  are the eigenvalues of  $A(\zeta)$  and, hence,

$$(5.23) \quad \Theta_{A(\zeta)} = (\lambda_1^{(\xi)}(\zeta) - A(\zeta)) \cdot \dots \cdot (\lambda_m^{(\xi)}(\zeta) - A(\zeta)).$$

<sup>6</sup>For the definition of a *normal* complex space, see, e.g., [L, Ch. VI, §2]. For example, each complex manifold is normal

In particular this shows that (5.22) is holomorphic on  $X \setminus \text{split } A$ .

Moreover, as  $|\lambda_j(\zeta)| \leq \|A(\zeta)\|$ , from (5.23) it follows that

$$(5.24) \quad \|\Theta_{A(\zeta)}\| \leq 2^m \|A(\zeta)\|^m \quad \text{for all } \zeta \in X \setminus \text{split } A.$$

Since  $X \cap \text{split } A$  is a nowhere dense analytic subset of  $X$ , and  $X$  is normal, this implies that (5.22) extends holomorphically to  $X$ . We denote this extended map by  $\Theta$ . By (5.24), then

$$(5.25) \quad \|\Theta(\zeta)^k\| \leq 2^{mk} \|A(\zeta)\|^{mk} \quad \text{for all } \zeta \in X \text{ and } 1 \leq k \leq n.$$

Set

$$r_k = \max_{\zeta \in X} \text{rank } \Theta(\zeta)^k \quad \text{for } 1 \leq k \leq n.$$

*First case:*  $r_1 = 0$ . Then  $(\Theta_{A(\zeta)})^k = 0$  for all  $\zeta \in X \setminus \text{split } A$  and  $k \in \mathbb{N}^*$ . In particular, each  $\xi \in X \setminus \text{split } A$  satisfies condition (iii) in Lemma 5.3. Hence  $X \setminus \text{Jst } A = \text{split } A$ , and the claim of the theorem follows from Theorem 4.3.

*Second case:*  $r_1 > 0$ . Then, by (5.2),  $n \geq 2$  and there is an integer  $1 \leq k_0 \leq n-1$  with  $r_{k_0} > 0$  and  $r_{k_0+1} = 0$ . For  $1 \leq k \leq k_0$ , let  $f_1^{(k)}, \dots, f_{s_k}^{(k)}$  be the minors of order  $r_k$  of  $\Theta^k$  which do not vanish identically on  $X$ . Since  $X$  is irreducible (i.e., the manifold of smooth points of  $X$  is connected), and the functions  $f_j^{(k)}$  are holomorphic and  $\neq 0$ , none of them can vanish identically on an open subset of  $X$ . Hence,

$$(5.26) \quad Z := \bigcup_{k=1}^{k_0} \{f_1^{(k)} = \dots = f_{s_k}^{(k)} = 0\}$$

is a nowhere dense analytic subset of  $X$ , and  $\xi \in Z$  if and only if  $\xi$  is a jump point (Def. 3.6) for at least one of the maps  $\Theta^1, \dots, \Theta^{k_0}$ . Since  $\Theta^k \equiv 0$  if  $k_0 + 1 \leq k \leq n-1$ , the latter means that  $\xi \in Z$  if and only if  $\xi$  is a jump point for at least one of the maps  $\Theta^1, \dots, \Theta^{n-1}$ . In particular,  $\xi \in Z \cap (X \setminus \text{split } A)$  if and only if  $\xi \in (X \setminus \text{split } A)$  and  $\xi$  is a jump point of at least one of the maps

$$X \setminus \text{split } A \mapsto (\Theta_{A(\zeta)})^1, \quad \dots, \quad X \setminus \text{split } A \mapsto (\Theta_{A(\zeta)})^{n-1},$$

i.e.,  $\xi \in Z \cap (X \setminus \text{split } A)$  if and only if  $\xi \in X \setminus \text{split } A$  and  $\xi$  violates condition (iii) in Lemma 5.3. Hence

$$(X \setminus \text{Jst } A) \cap (X \setminus \text{split } A) = Z \cap (X \setminus \text{split } A).$$

Since  $\text{split } A \subseteq X \setminus \text{Jst } A$ , it follows that

$$(5.27) \quad X \setminus \text{Jst } A = Z \cup \text{split } A.$$

By Theorem 4.3, we have finitely many holomorphic functions  $g_1, \dots, g_p : X \rightarrow \mathbb{C}$ , each of which is a finite sum of finite products of elements of  $A$ , such that

$$(5.28) \quad \text{split } A = \{g_1 = \dots = g_p = 0\},$$

and

$$(5.29) \quad |g_j(\zeta)| \leq (2n)^{6n^2} \|A(\zeta)\|^{2n^2} \quad \text{for all } \zeta \in X \text{ and } 1 \leq j \leq p.$$

Now let  $\{h_1, \dots, h_\ell\}$  be the set of all functions of the form

$$g_j \cdot \prod_{k=1}^{k_0} f_{\kappa_k}^{(k)}$$

with  $1 \leq j \leq p$  and  $1 \leq \kappa_k \leq s_k$  for  $1 \leq k \leq k_0$ . Then (5.20) follows from (5.26), (5.27) and (5.28).

By (5.25), for all  $1 \leq k \leq k_0$ ,  $1 \leq j \leq s_k$  and  $\zeta \in X$ , we have

$$|f_j^{(k)}(\zeta)| \leq r_k! 2^{mk\tau_k} \|A(\zeta)\|^{mk\tau_k}.$$

Together with (5.29) this yields estimate (5.21) (recall that  $n \geq 2$ ).

Next we consider the case when  $X$  is irreducible, but, possibly, not normal.

Let  $\pi : \tilde{X} \rightarrow X$  be the normalization of  $X$  (see, e.g., [L, Ch. VI, §4]) and  $\tilde{A} := A \circ \pi$ . Then  $\tilde{X}$  is normal and irreducible. Therefore, by part (i) of the theorem,  $\tilde{X} \setminus \text{Jst } \tilde{A}$  is a nowhere dense closed analytic subset of  $\tilde{X}$ . Since, clearly,

$$(5.30) \quad \pi(\tilde{X} \setminus \text{Jst } \tilde{A}) = X \setminus \text{Jst } A,$$

this implies, by Remmert's proper mapping theorem (see, e.g., [L, Ch. V, §5.1]), that  $X \setminus \text{Jst } A$  is a closed analytic subset of  $X$ .

To prove that  $X \setminus \text{Jst } A$  is nowhere dense in  $X$ , let  $X^0$  be the manifold of smooth points of  $X$ . Then  $\pi$  is biholomorphic between  $\pi^{-1}(X^0)$  and  $X^0$ , and, by (5.30),

$$\pi(\pi^{-1}(X^0) \setminus \text{Jst } \tilde{A}) = X^0 \setminus \text{Jst } A.$$

Since  $\pi^{-1}(X^0) \setminus \text{Jst } \tilde{A}$  is nowhere dense in  $\pi^{-1}(X^0)$ , this implies that  $X^0 \setminus \text{Jst } A$  is nowhere dense in  $X^0$ . Since  $X \setminus X^0$  is nowhere dense in  $X$ , it follows that  $X \setminus \text{Jst } A$  is nowhere dense in  $X$ .

Finally, we consider the general case.

By the global decomposition theorem for complex spaces (see, e.g., [L, V.4.6] or [GR, Ch. 9, §2.2]), there is a locally finite covering  $\{X_i\}_{i \in I}$  of  $X$  such that each  $X_i$  is an irreducible closed analytic subset of  $X$ . Then, as already proved, each  $X_i \setminus \text{Jst } (A|_{X_i})$  is a nowhere dense analytic subset of  $X_i$ . Since the covering  $\{X_i\}_{i \in I}$  is locally finite and, clearly,

$$X \setminus \text{Jst } A = \bigcup_{i \in I} (X_i \setminus \text{Jst } (A|_{X_i})),$$

this proves that  $X \setminus \text{Jst } A$  is a nowhere dense analytic subset of  $X$ .  $\square$

**5.6. Remark.** Estimate (5.21) shows that the claim of Theorem 5.5 can be completed. For example:

– If  $A$  is bounded, then  $X \setminus \text{Jst } A$  can be defined by bounded holomorphic functions. In the case of the disk  $X = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$  this implies that  $X \setminus \text{Jst } A$  satisfies the Blaschke condition.

– If  $X = \mathbb{C}^N$  and the elements of  $A$  are holomorphic polynomials, then  $\mathbb{C}^N \setminus \text{Jst } A$  is the common zero set of finitely many holomorphic polynomials, i.e., it is affine algebraic. For  $N = 1$  this means that  $\mathbb{C} \setminus \text{Jst } A$  is finite.

**5.7. Remark.** It is possible (in contrast to Remark 4.4) that the set of points which are not Jordan stable is of codimension  $> 1$ , also at smooth points. Here is an example. Let

$$A(z, w) := \begin{pmatrix} zw & -z^2 \\ w^2 & -zw \end{pmatrix} \quad \text{for } (z, w) \in \mathbb{C}^2.$$

Then  $A(z, w)^2 = 0$  for all  $(z, w) \in \mathbb{C}^2$ , and  $A(z, w) = 0$  if and only if  $(z, w) = 0$ . This means that  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  is the Jordan normal form of  $A(0, 0)$ , whereas, for all

$(z, w) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is the Jordan normal form of  $A(z, w)$ . Hence,  $(0, 0)$  is the only point in  $\mathbb{C}^2$  which is not Jordan stable for  $A$ .

## 6. CONTINUOUS BOUNDARY VALUES

In Sections 4 and 5, we described the set of Jordan stable points for holomorphic matrices. Corresponding results can be obtained also for certain other classes of continuous matrices. In the present section we consider the following example.

Let  $X$  be a connected open set of some  $\mathbb{C}^N$  and let  $\overline{X}$  be the closure of  $X$  in  $\mathbb{C}^N$ . Assume that the boundary of  $X$ ,  $\partial X := \overline{X} \setminus X$ , is  $\mathcal{C}^0$  smooth.<sup>7</sup> Let  $A : \overline{X} \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be continuous on  $\overline{X}$ , and holomorphic in  $X$ . Denote by  $\text{split } A$  the set of splitting points of the eigenvalues of  $A$  (Def. 4.1), and by  $\text{Jst } A$  the set of Jordan stable points of  $A$  (Def. 5.4).

Then it is easy to check that only minor modifications of the proof of Theorem 4.3 (and the proofs of the results used in this proof) are necessary to obtain the following

**6.1. Theorem.** *If  $\text{split } A \neq \emptyset$ , then there exist finitely many continuous on  $\overline{X}$  and holomorphic in  $X$  functions  $h_1, \dots, h_\ell : \overline{X} \rightarrow \mathbb{C}$ , each of which is a finite sum of finite products of elements of  $A$ , such that*

$$\text{split } A = \{h_1 = \dots = h_\ell = 0\}$$

and

$$(6.1) \quad |h_j(\zeta)| \leq (2n)^{6n^2} \|A(\zeta)\|^{2n^2} \quad \text{for all } \zeta \in \overline{X} \text{ and } 1 \leq j \leq \ell.$$

To describe the full set  $X \setminus \text{Jst } A$ , we need some preparation.

**6.2. Lemma and Definition.** *Let  $\xi \in \text{split } A$ , let  $\mu_1, \dots, \mu_k$  be the eigenvalues of  $A(\xi)$ , and set  $\mathbb{D}(\mu_j, \varepsilon) := \{\lambda \in \mathbb{C} \mid |\lambda - \mu_j| < \varepsilon\}$  for  $1 \leq j \leq k$ . Then there exist positive integers  $\kappa_1, \dots, \kappa_k$  such that the following holds:*

*If  $\varepsilon > 0$  and the disks  $\mathbb{D}(\mu_1, \varepsilon), \dots, \mathbb{D}(\mu_k, \varepsilon)$  are pairwise disjoint, then there exists a neighborhood  $U_\xi$  in  $\overline{X}$  of  $\xi$  such that*

- (i) *for each  $\zeta \in U_\xi$ , all eigenvalues of  $A(\zeta)$  lie in  $D(\mu_1, \varepsilon) \cup \dots \cup D(\mu_k, \varepsilon)$ ,*
- (ii) *for each  $\zeta \in U_\xi \setminus \text{split } A$  and each  $1 \leq j \leq k$ , exactly  $\kappa_j$  of the eigenvalues of  $A(\zeta)$  lie in  $\mathbb{D}(\mu_j, \varepsilon)$ .*

*We call  $\kappa_j$  the **splitting amount** of  $\mu_j$ .*

*Proof.* Let  $\varepsilon > 0$  be given such that the disks  $\mathbb{D}(\mu_1, \varepsilon), \dots, \mathbb{D}(\mu_k, \varepsilon)$  are pairwise disjoint.

Let  $P_A(\zeta)(\lambda) := \det(\lambda - A(\zeta))$  be the characteristic polynomial of  $A(\zeta)$ . Since  $A$  is continuous and no zeros of  $P_A(\xi)$  lie on  $\partial \mathbb{D}(\mu_1, \varepsilon) \cup \dots \cup \partial \mathbb{D}(\mu_k, \varepsilon)$ , we can find a neighborhood  $U_\xi$  in  $\overline{X}$  of  $\xi$  so small that

$$(6.2) \quad |P_A(\zeta)(\lambda) - P_A(\xi)(\lambda)| < |P_A(\xi)(\lambda)|$$

for all  $\zeta \in U_\xi$  and  $\lambda \in \partial \mathbb{D}(\mu_1, \varepsilon) \cup \dots \cup \partial \mathbb{D}(\mu_k, \varepsilon)$ .

Since  $\partial X$  is  $\mathcal{C}^0$  smooth, we may moreover assume that  $U_\xi \cap X$  is connected. As  $\text{split } A$  is a nowhere dense analytic subset of  $X$  (Theorem 4.3), it follows that  $(U_\xi \cap X) \setminus \text{split } A$  is connected. Since  $\overline{X} \setminus \text{split } A$  is open in  $\overline{X}$  (by Theorem 6.1)

<sup>7</sup>By that we mean that, for each  $\xi \in \partial X$ , there is a neighborhood of  $\xi$  in  $\overline{X}$  which is homeomorphic to a set of the form  $U \cap \{(x_1, \dots, x_{2N}) \in \mathbb{R}^{2N} \mid x_1 \geq 0\}$ , where  $U$  is an open subset of  $\mathbb{R}^{2N}$ .



and since  $\partial D$  is  $\mathcal{C}^0$  smooth, the connectedness of  $(U_\xi \cap X) \setminus \text{split } A$  implies that also  $U_\xi \setminus \text{split } A$  is connected.

Let  $\nu_j$  be the algebraic multiplicity of  $\mu_j$  as an eigenvalue of  $A(\xi)$ , i.e., its order as a zero of  $P_A(\xi)$ . Then

$$(6.3) \quad \nu_1 + \dots + \nu_k = n.$$

From (6.2) it follows, by Rouché's theorem, that, for each  $\zeta \in U_\xi$  and each  $1 \leq j \leq k$ , counting multiplicities, exactly  $\nu_j$  zeros of  $P_A(\zeta)$  lie in  $\mathbb{D}(\mu_j, \varepsilon)$ . By (6.3) this implies that, for each  $\zeta \in U_\xi$ , counting multiplicities, exactly  $n$  zeros of  $P_A(\zeta)$  lie in  $\mathbb{D}(\mu_1, \varepsilon) \cup \dots \cup \mathbb{D}(\mu_k, \varepsilon)$ . Since the degree of  $P_A(\zeta)$  is  $n$ , this proves (i).

To prove (ii), for  $\zeta \in U_\xi \setminus \text{split } A$  and  $1 \leq j \leq k$ , we denote by  $\kappa_j(\zeta)$  the number of eigenvalues of  $A(\zeta)$  in  $\mathbb{D}(\mu_j, \varepsilon)$ , not counting multiplicities. We have to prove that the functions

$$(6.4) \quad U_\xi \setminus \text{split } A \ni \zeta \longmapsto \kappa_j(\zeta)$$

are constant. Since  $U_\xi \setminus \text{split } A$  is connected, it is sufficient to prove that these maps are locally constant.

For that, fix  $\zeta_0 \in U_\xi \setminus \text{split } A$ , and let  $w_s^{(j)}$ ,  $1 \leq s \leq \kappa_j(\zeta_0)$ , be the eigenvalues of  $A(\zeta_0)$  which lie in  $\mathbb{D}(\mu_j, \varepsilon)$ ,  $1 \leq j \leq k$ . Choose  $\delta > 0$  so small that the disks

$$\mathbb{D}(w_s^{(j)}, \delta) := \left\{ \lambda \in \mathbb{C} \mid |\lambda - w_s^{(j)}| < \delta \right\}, \quad 1 \leq j \leq k, \quad 1 \leq s \leq \kappa_j(\zeta_0),$$

are pairwise disjoint, and

$$(6.5) \quad \mathbb{D}(w_s^{(j)}, \delta) \subseteq \mathbb{D}(\mu_j, \varepsilon) \quad \text{for all } 1 \leq j \leq k \text{ and } 1 \leq s \leq \kappa_j(\zeta_0).$$

Then, by Lemma 4.2, there exists a neighborhood  $V_{\zeta_0} \subseteq U_\xi \setminus \text{split } A$  of  $\zeta_0$  and continuous functions

$$\lambda_s^{(j)} : V_{\zeta_0} \longrightarrow \mathbb{D}(w_s^{(j)}, \delta)$$

such that, for all  $\zeta \in V_{\zeta_0}$ ,

$$\lambda_s^{(j)}(\zeta), \quad 1 \leq j \leq k, \quad 1 \leq s \leq \kappa_j(\zeta_0),$$

are the eigenvalues of  $A(\zeta)$ . Since the disks  $\mathbb{D}(w_s^{(j)}, \delta)$  are pairwise disjoint, and by (6.5), this in particular means that  $\kappa_j(\zeta) = \kappa_j(\zeta_0)$  for all  $\zeta \in V_{\zeta_0}$  and all  $1 \leq j \leq k$ . Hence, the functions (6.4) are constant in a neighborhood of  $\zeta_0$ .  $\square$

**6.3. Definition.** For  $\zeta \in \overline{X}$ , let  $\lambda_1(\zeta), \dots, \lambda_{m(\zeta)}(\zeta)$  be the eigenvalues of  $A$ , and let  $\kappa_1(\zeta), \dots, \kappa_{m(\zeta)}(\zeta)$  be the splitting amounts of  $\lambda_1(\zeta), \dots, \lambda_{m(\zeta)}(\zeta)$ , respectively. Then, for  $\zeta \in \overline{X} \setminus \text{split } A$ , we define

$$\Theta_A(\zeta) = \begin{cases} (\lambda_1(\zeta) - A(\zeta)) \cdot \dots \cdot (\lambda_{m(\zeta)}(\zeta) - A(\zeta)) & \text{if } \zeta \in \overline{X} \setminus \text{split } A, \\ (\lambda_1(\zeta) - A(\zeta))^{\kappa_1(\zeta)} \cdot \dots \cdot (\lambda_{m(\zeta)}(\zeta) - A(\zeta))^{\kappa_{m(\zeta)}(\zeta)} & \text{if } \zeta \in \text{split } A. \end{cases}$$

**6.4. Lemma.** *The function*

$$(6.6) \quad \overline{X} \ni \zeta \longmapsto \Theta_A(\zeta)$$

*is continuous on  $\overline{X}$ , and holomorphic in  $X$ .*

*Proof.* By Lemma 4.2, for each  $\xi \in \overline{X} \setminus \text{split } A$ , we have a neighborhood  $U$  in  $\overline{X} \setminus \text{split } A$  of  $\xi$ , and functions  $\lambda_1, \dots, \lambda_m : U \rightarrow \mathbb{C}$  (uniquely determined up to the

order), which are continuous on  $U$  and holomorphic in  $U \cap X$ , such that, for each  $\zeta \in U$ ,  $\lambda_1(\zeta), \dots, \lambda_m(\zeta)$  are the eigenvalues of  $A(\zeta)$  and, therefore,  $m = m(\zeta)$  and

$$\Theta_{A(\zeta)} = (\lambda_1(\zeta) - A(\zeta)) \cdot \dots \cdot (\lambda_m(\zeta) - A(\zeta)) \quad \text{for } \zeta \in U.$$

This shows that (6.6) is continuous on  $\overline{X} \setminus \text{split } A$ , and holomorphic in  $X \setminus \text{split } A$ .

It remains to prove that (6.6) is continuous at each point of  $\text{split } A$ . The holomorphy on  $X$  then follows from the Riemann extension theorem, as  $X \cap \text{split } A$  is a nowhere dense closed analytic subset of  $X$  (by Theorem 4.3).

Since  $X \cap \text{split } A$  is nowhere dense in  $X$  and  $\partial X$  is  $\mathcal{C}^0$  smooth, it follows that  $\text{split } A$  is nowhere dense in  $\overline{X}$ . Therefore it is sufficient to prove that, for each  $\xi \in \text{split } A$  and each  $\varepsilon > 0$ , there exists a neighborhood  $U_\xi$  in  $\overline{X}$  of  $\xi$  such that

$$(6.7) \quad \|\Theta_A(\zeta) - \Theta_A(\xi)\| < \varepsilon \quad \text{if } \zeta \in U_\xi \setminus \text{split } A.$$

Let  $\xi \in \text{split } A$  and  $\varepsilon > 0$  be given.

Let  $\mu_1, \dots, \mu_k$  be the eigenvalues of  $A(\xi)$ , and let  $\kappa_j$  be the splitting amount of  $\mu_j$ . Since  $A$  is continuous, then we can find a neighborhood  $U_\xi$  in  $\overline{X}$  of  $\xi$ , and  $\delta > 0$  so small that the disks  $\mathbb{D}(\mu_j, \delta) := \{\lambda \in \mathbb{C} \mid |\lambda - \mu_j| < \delta\}$ ,  $1 \leq j \leq k$ , are pairwise disjoint, and, for all  $\zeta \in U_\xi$  and all  $z_i^{(j)} \in \mathbb{D}(\mu_j, \delta)$ ,  $1 \leq j \leq k$ ,  $1 \leq i \leq \kappa_j$ ,

$$\left\| \prod_{j=1}^k \prod_{i=1}^{\kappa_j} (z_i^{(j)} - A(\zeta)) - \prod_{j=1}^k (\mu_j - A(\xi))^{\kappa_j} \right\| < \varepsilon,$$

and, hence,

$$(6.8) \quad \left\| \prod_{j=1}^k \prod_{i=1}^{\kappa_j} (z_i^{(j)} - A(\zeta)) - \Theta_A(\xi) \right\| < \varepsilon.$$

By Lemma 6.2, shrinking  $U_\xi$ , we can moreover achieve that

- for each  $\zeta \in U_\xi$ , all eigenvalues of  $A(\zeta)$  lie in  $D(\mu_1, \varepsilon) \cup \dots \cup D(\mu_k, \delta)$ , and
- for each  $\zeta \in U_\xi \setminus \text{split } A$  and each  $1 \leq j \leq k$ , exactly  $\kappa_j$  of the eigenvalues of  $A(\zeta)$ , which we denote by  $\lambda_1^{(j)}(\zeta), \dots, \lambda_{\kappa_j}^{(j)}(\zeta)$ , lie in  $\mathbb{D}(\mu_j, \delta)$ .

Then, by (6.8), for all  $\zeta \in U_\xi \setminus \text{split } A$ ,

$$\left\| \prod_{j=1}^k \prod_{i=1}^{\kappa_j} (\lambda_i^{(j)}(\zeta) - A(\zeta)) - \Theta(\xi) \right\| < \varepsilon.$$

where, by definition of  $\Theta(\zeta)$ ,

$$\prod_{j=1}^k \prod_{i=1}^{\kappa_j} (\lambda_i^{(j)}(\zeta) - A(\zeta)) = \Theta(\zeta).$$

which proves (6.7).  $\square$

By this lemma, slightly modifying the proof of Theorem 5.5, one obtains

**6.5. Theorem.**  *$X \setminus \text{Jst } A$  is a nowhere dense analytic subset of  $X$ ,  $\overline{X} \setminus \text{Jst } A$  is nowhere dense in  $\overline{X}$ , and, if  $\text{Jst } A \neq \overline{X}$ , then there exist finitely many functions  $h_1, \dots, h_\ell : \overline{X} \rightarrow \mathbb{C}$ , which are continuous on  $\overline{X}$ , and holomorphic in  $X$ , such that*

$$(6.9) \quad \overline{X} \setminus \text{Jst } A = \{h_1 = \dots = h_\ell = 0\},$$

and

$$(6.10) \quad |h_j(\zeta)| \leq (2n)^{2n^4} \|A(\zeta)\|^{2n^4} \quad \text{for all } \zeta \in \overline{X} \text{ and } 1 \leq j \leq \ell.$$

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